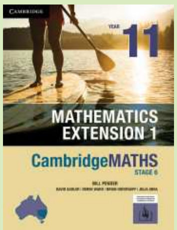
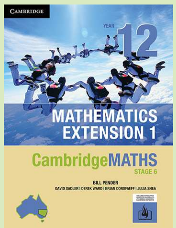
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1/2
(YEAR 11/12 COURSE)

Topic summary and exercises:

With references to





(X1) (X2) **Further Trigonometric Identities 2**

Name: ....

Initial version by H. Lam, April 2020. Last updated October 22, 2022.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

⚠ Beware! Heed warning.

(2) Mathematics content.

(x1) Mathematics Extension 1 content.

(x2) Mathematics Extension 2 content.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-3 applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems

ME12-4 uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

ME12-6 chooses and uses appropriate technology to solve problems in a range of contexts

Syllabus subtopics

ME-T2 Further Trigonometric Identities

ME-T3 Trigonometric Equations

ME-C2 Further Calculus Skills

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) or *CambridgeMATHS Year 12 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Trigonometric identities

1.1 Compound and double angle formulae

Learning Goal(s)

Knowledge
Trigonometric identities from Year 11 Course

Skills
Manipulation of identities in both directions

Understanding
Forward and backward directions of the identities

By the end of this section am I able to:

26.1 Consolidate trigonometric identities from Year 11.

26.2 Prove and apply other trigonometric identities, for example, $\cos 3x = 4\cos^3 x - 3\cos x$.

Laws/Results

Compound angle formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Laws/Results

Double angle formulae Replace B with A in the previous 'addition' formulae:

$$\sin 2A = \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A$$
$$\cos 2A = \cos A\cos A - \sin A\sin A$$
$$= \cos^2 A - \sin^2 A \qquad \text{(Variant 1)}$$
$$= 2\cos^2 A - 1 \qquad \text{(Variant 2)}$$
$$= 1 - 2\sin^2 A \qquad \text{(Variant 3)}$$
$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$
$$= \frac{2\tan A}{1 - \tan^2 A}$$

Example 1

[2007 VCE Specialist Mathematics Paper 1 Q10] (3 marks)

Given that $\tan 2x = \dfrac{4\sqrt{2}}{7}$ where $x \in \left[0, \dfrac{\pi}{4}\right)$, find the exact value of $\sin x$. **Answer:** $\frac{1}{3}$

Example 2

[**Ex 17D Q10**] Use compound angle formulae to find the exact value of

$$\cos\left(\tan^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{4}\right)$$

Example 3

[2008 Ext 1 HSC Q6] (3 marks) It can be shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all values of θ (Do NOT prove this.)

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin\theta$ for $0 \leq \theta \leq 2\pi$.

Example 4

[2022 CSSA Ext 1 Q12] (3 marks) Givne the pair of parametric equations

$$\begin{cases} x = 3\cos^2\theta \\ y = \sin\theta\cos\theta \end{cases}$$

for $\theta \in \left[0, \frac{\pi}{2}\right]$, find a Cartesian equation in the form $y = f(x)$. **Answer:** $y = \frac{1}{3}\sqrt{3x - x^2}$

Further exercises

Ex 17D (Pender et al., 2019a)

- Q10-12, 18-19

Ex 17E (Pender et al., 2019a)

- Q6-8, 13, 15

1.1.1 Further questions

1. [2006 VCE Specialist Mathematics Paper 2, Section 2 Q5]

(b) Use a double angle formula to show that the exact value of **3**

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Explain why any values are rejected.

(c) Hence show that the exact value of $\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$. **2**

1.2 Products to sums

Learning Goal(s)

Knowledge
Products to sum formulae

Skills
Manipulate compound angle formulae to arrive at products to sum formulae

Understanding
Where the products to sum formulae arise from

By the end of this section am I able to:

26.3 Derive and use the formulae for trigonometric products as sums and differences for $\cos A \cos B$, $\sin A \sin B$, $\sin A \cos B$ and $\cos A \sin B$.

Laws/Results

- These results arise from the sine/cosine compound angle formulae.
- Also available on the Reference Sheet.
- Generally, $A > B$. Select the appropriate formula for your convenience.

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

1.2.1 Derivation of $\cos A \cos B$ and $\sin A \sin B$

Steps

1. Write the cosine compound angle formulae: $\cos(A - B)$ and $\cos(A + B)$.

2. Add these together, and change the subject to $\cos A \cos B$:

3. Subtract these, and change the subject to $\sin A \sin B$:

1.2.2 Derivation of $\sin A \cos B$ and $\cos A \sin B$

Steps

1. Write the sine compound angle formulae: $\sin(A+B)$ and $\sin(A-B)$.

2. Add these together, and change the subject to $\sin A \cos B$:

3. Subtract these, and change the subject to $\cos A \sin B$:

Important note

These formulae should be used in *both directions*, i.e. left to right and right to left.

1.2.3 Equations and identities

Example 5

[Ex 4.5] (Fitzpatrick & Aus, 2019)

1. Show that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$.

2. Show that $\dfrac{\sin A + \sin(A+B) + \sin(A+2B)}{\cos A + \cos(A+B) + \cos(A+2B)} = \tan(A+B)$.

Example 6

[Ex 4.8] (Fitzpatrick & Aus, 2019)
Solve $\sin 2\theta \cos\theta = \sin 3\theta \cos 2\theta$ for $\theta \in [0, \pi]$.

Answer: $0,\ \frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{5\pi}{8},\ \frac{7\pi}{8},\ \pi$

Example 7

[1995 4U HSC Q5]

i. Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

ii. Hence or otherwise, find all solutions of $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **3**

Answer: $0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, 2\pi$

Example 8

[2007 Ext 2 HSC Q4]

i. Show that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$. **2**

ii. Show that **2**

$$4\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)=\sin 3\theta$$

iii. Write down the maximum value of **1**

$$\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)$$

Example 9

[2021 Ext 1 HSC Q13]

i. The numbers A, B and C are related by the equation $A = B - d$ and $C = B + d$, where d is a constant. **2**

Show that $\dfrac{\sin A + \sin C}{\cos A + \cos C} = \tan B$.

ii. Hence or otherwise, solve $\dfrac{\sin \frac{5\theta}{7} + \sin \frac{6\theta}{7}}{\cos \frac{5\theta}{7} + \cos \frac{6\theta}{7}} = \sqrt{3}$ for $0 \le \theta \le 2\pi$ **2**

Answer: $\theta = \frac{14\pi}{33}, \frac{56\pi}{33}$

Important note

Remember the product to sums formulae can also be used in reverse.

Example 10

[2006 Ext 2 HSC Q5]

i. Show that $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$. **1**

ii. Hence or otherwise, solve the equation **3**

$$\cos\theta+\cos 2\theta+\cos 3\theta+\cos 4\theta=0$$

for $0\leq\theta\leq 2\pi$.

Answer: $\frac{\pi}{5},\frac{\pi}{2},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{3\pi}{2},\frac{9\pi}{5}$

1.2.4 Integrals

Example 11

[2003 Ext 2 HSC Q6]

i. Prove the identity $\cos(a+b)x + \cos(a-b)x = 2\cos ax \cos bx$. **1**

ii. Hence find $\displaystyle\int \cos 3x \cos 2x \, dx$. **2**

Answer: $\frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

Example 12

Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sin 2\theta \cos 3\theta \, d\theta$. **Answer:** $\frac{3\sqrt{3}-4}{10}$

Example 13

[Ex 7E Q20] (Pender et al., 2019b)

(a) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.

(b) Hence find:

i. $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin 3x\cos 2x\,dx$

ii. $\displaystyle\int_0^{\pi} \sin 3x\cos 4x\,dx$

(c) Show that $\displaystyle\int_{-\pi}^{\pi} \sin mx\cos nx\,dx=0$ for positive integers m and n:

i. using the primitive

ii. using symmetry arguments

Further exercises

Ex 17G (Pender et al., 2019a) - Section 1.2 only

- Q1(b), 2-8

Ex 11A (Pender et al., 2019b) - all of Section 1.

- All questions

Section 2

Further techniques for solving trigonometric equations

2.1 t formulae

Learning Goal(s)

Knowledge
When to use the t formulae

Skills
Manipulate expressions and prove identities using the t formulae

Understanding
Where the t formulae arise from

By the end of this section am I able to:

26.4 Derive and use expressions for $\sin A$, $\cos A$ and $\tan A$ in terms of t where $t = \tan\frac{A}{2}$.

26.6 Solve trigonometric equations requiring factorising and/or the application of compound angle, double angle formulae or the t formulae

26.7 Solve trigonometric equations and interpret solutions in context using technology or otherwise

Definition 1

t formulae If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

2.1.1 Derivation

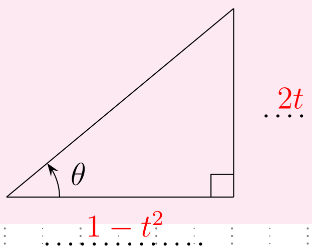
Steps

1. If $t = \tan\frac{\theta}{2}$, then

$$\tan\theta = \tan\left(2 \times \frac{\theta}{2}\right) = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2t}{1-t^2}.$$

2. Draw a right angled triangle with angle θ, use Pythagoras' Theorem to find the missing length of the hypotenuse and obtain expressions for $\sin\theta$ and $\cos\theta$ in terms of t:



Example 14

Use the t formulae to prove

(a) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$.

(b) $\sec 2x + \tan 2x = \tan\left(x + \dfrac{\pi}{4}\right)$

Important note

If a question only contains a trigonometric term involving a single angle and its equivalent double angle, e.g. $\sin\theta$ and $\sin 2\theta$, use $t = \tan\theta$ instead of $t = \tan\frac{\theta}{2}$.

Further exercises

Ex 17F (Pender et al., 2019a)

- Q1-10

2.1.2 Solution of equations

Steps

1. Use transformation provided to convert equation with trigonometric terms into a ....polynomials........ in t.
2. Solve via ....polynomial........ techniques.
3. Transform solution back into θ.

Important note

Do *not* apply the t-formulae unless asked to do so.

Example 15

[**Ex 2B Q6**] (Pender, Sadler, Shea, & Ward, 2000)

(a) Given that $t = \tan 112.5°$, show that $\dfrac{2t}{1-t^2} = 1$.

(b) i. Hence show that $\tan 112.5° = -\sqrt{2} - 1$.

ii. What does the other root of the equation represent?

Example 16

[Ex 8.3] (Fitzpatrick & Aus, 2019) Solve

$$5\cos\theta - 2\sin\theta = 2$$

for $0^\circ \le \theta \le 360^\circ$ by using the substitution $t = \tan\frac{\theta}{2}$. **Answer:** $46^\circ 24'$, 270°

Example 17

If α and β are roots of the equation

$$\cos x + 3\sin x + 2 = 0$$

and $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are not equal, find the value of

$$\tan^2\frac{\alpha}{2} + \tan^2\frac{\beta}{2}$$

Answer: 30

Example 18

Solve $7\sin x - 4\cos x = 4$ for $0° \le x \le 360°$ by using the substitution $t = \tan\frac{x}{2}$.

Important note

Asymptotes for $\tan\frac{\theta}{2}$

- $t = \tan\frac{\theta}{2}$ will not produce solutions where $\theta = \pi$.
- This occurs when t^2 terms cancel out, or when the coefficient of $\cos\theta$ is equal in magnitude but opposite in sign to that of the constant term.
- A more reliable method is the *auxiliary angle method*, starting in Section 2.2 on page 30

Example 19

[2009 Ext 2 HSC Q8] ⚠

i. Using the substitution $t = \tan\frac{\theta}{2}$ or otherwise, show that **2**

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}$$

ii. Use mathematical induction to prove that, for integers $n \geq 1$, **3**

$$\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$$

iii. Show that **2**

$$\lim_{n\to\infty}\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{2}{x} - 2\cot x$$

iv. Hence find the exact value of **2**

$$\tan\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{8} + \frac{1}{4}\tan\frac{\pi}{16} + \cdots$$

Further exercises

Ex 11C (Pender et al., 2019b)

- Q1-10

2.2 Auxiliary angle method

Learning Goal(s)

Knowledge
When to use the auxiliary angle method

Skills
Deriving the results needed to find the auxiliary angle and solve problems

Understanding
Where the auxiliary angle method arises from

By the end of this section am I able to:

26.5 Convert expressions of the form $a\cos x + b\sin x$ to $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$ and apply these to solve equations of the form $a\cos x + b\sin x = c$, sketch graphs and solve related problems

2.2.1 Sum of two sinusoids with the same frequency

GeoGebra

Investigate altering the a, b and ω_1 values in this GeoGebra applet.

- `https://www.geogebra.org/m/vthmx6nt`

Laws/Results

For the sum of two sinusoidal terms:

$$y = a\sin nx + b\cos nx$$

where $a,\ b \in \mathbb{R}$ and $n \in \mathbb{R}^+$, the resultant curve when adding $a\sin nx$ with $b\cos nx$ will result in:

- altered ......amplitude......
- altered ......phase......
- unaltered ......frequency......

$y = a\sin nx + b\cos nx$ can be ......'collapsed'...... into new a single sinusoidal term with a new ......amplitude...... and ......phase...... ......shift...... :

$$y = R\sin(nx \pm \alpha) \qquad \text{or} \qquad y = R\cos(nx \pm \alpha)$$

for some R and $\alpha \in \mathbb{R}$.

Definition 2

The *auxiliary angle* is α from the above result.

Definition 3

The *auxiliary angle method* is the process to convert the sum of two sinusoidal terms with the same ......frequency...... , into a new sinusoidal term with a new ......amplitude...... and an additional *auxiliary angle*.

2.2.2 Derivation

Steps

For a, $b > 0$ and the following sums of sinusoids, use the sine and cosine compound angle formulae to equate 'coefficients':

- $a \sin x \pm b \cos x$: choose $R\sin(x \pm \alpha)$

$$a \sin x \pm b \cos x \equiv R\sin(x \pm \alpha)$$
$$= R\sin x \cos\alpha \pm R\cos x \sin\alpha$$

Equating coefficients in $\sin x$ and $\cos x$:

$$\therefore R\cos\alpha = a \qquad R\sin\alpha = b$$

- $a \cos x \pm b \sin x$: choose $R\cos(x \mp \alpha)$

$$a \cos x \pm b \sin x \equiv R\cos(x \mp \alpha)$$
$$= R\cos x \cos\alpha \mp R\sin x \sin\alpha$$

Equating coefficients in $\sin x$ and $\cos x$:

$$\therefore R\cos\alpha = a \qquad R\sin\alpha = b$$

Then use other techniques for solving pairs of simultaneous equations to obtain values for R and α.

Important note

Choose the correct sinusoidal term so that R and α are minimised and positive, i.e.:

- $R > 0$.
- $0 < \alpha < \frac{\pi}{2}$.

Important note

Do *not* memorise the result $R = \sqrt{a^2 + b^2}$ or $\tan\alpha = \frac{b}{a}$ that some textbooks try to provide. Memorise the technique instead.

2.2.3 Curve sketching

- After ......collapsing...... the sum of two sinusoidal terms, answer the question!

Example 20

[2003 Ext 1 HSC Q2]

i. Express $\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where α is in radians. **2**

ii. Hence, or otherwise, sketch the graph of $y = \cos x - \sin x$ for $0 \le x \le 2\pi$. **2**

Example 21

[2014 Sydney Grammar Ext 1 Trial Q12]

i. Write the expression $\sqrt{2}\sin x - \sqrt{6}\cos x$ in the form $A\sin(x-\theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$. **2**

ii. Hence write down the maximum value of $\sqrt{2}\sin x - \sqrt{6}\cos x$, and find the smallest positive value of x for which this maximum occurs. **2**

Example 22

[2020 Ext 1 HSC Sample Q13] (4 marks) A device playing a signal given by $x = \sqrt{2}\sin t + \cos t$ produces distortion whenever $|x| \geq 1.5$.

For what fraction of the time will the device produce distortion if the signal is played continuously? **Answer:** $\frac{1}{3}$

2.2.4 Solution of equations

- After ....collapsing........ the sum of two sinusoidal terms, answer the question!

Example 23

[2009 Ext 1 HSC Q2]

i. Express $3\sin x + 4\cos x$ in the form $A\sin(x+\alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$. **2**

ii. Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \le x \le 2\pi$. Give your answer, or answers, correct to two decimal places. **2**

Example 24

[2022 CSSA Ext 1 Q11]

i. Write the expression $2\sqrt{3}\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. **2**

ii. Hence find the value of k, where $0 \le k \le 2\pi$, for which **3**

$$\int_0^k \left(2\sqrt{3}\cos x - 2\sin x\right)\, dx = 2$$

Answer: $k = \frac{\pi}{3}$

Example 25

[2007 Ext 1 HSC Q6] A particle moves in a straight line. Its displacement, x metres, after t seconds is given by

$$x = \sqrt{3}\sin 2t - \cos 2t + 3$$

i. Moved to new Extension 2 syllabus.

ii. What is the period of the motion? **1**

iii. Express the velocity of the particle in the form **1**

$$\dot{x} = A\cos(2t - \alpha)$$

where α is in radians.

iv. Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction. **2**

Further exercises

Ex 11B (Pender et al., 2019b)

- Q1-19

2.2.5 Further questions

1. [2010 Ext 1 HSC Q2]

i. Express $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right)$ in the form $R\cos(\theta + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **3**

ii. Hence, or otherwise, solve **2**

$$2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right) = 3$$

for $0 < \theta < 2\pi$.

2.3 Substitutions and roots of polynomials

Example 26

[**2020 Ext 1 HSC Sample Q13/Ex 17E Q12**]

i. Prove the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **3**

ii. Hence find expressions for the exact values of the solutions to the equation $8x^3 - 6x = 1$. **4**

Example 27

[2020 Ext 1 HSC Q14]

i. Show that $\sin^3\theta - \dfrac{3}{4}\sin\theta + \dfrac{\sin(3\theta)}{4} = 0$. **2**

ii. By letting $x = 4\sin\theta$ in the cubic equation $x^3 - 12x + 8 = 0$, show that $\sin(3\theta) = \dfrac{1}{2}$. **2**

iii. Prove that $\sin^2\dfrac{\pi}{18} + \sin^2\dfrac{5\pi}{18} + \sin^2\dfrac{25\pi}{18} = \dfrac{3}{2}$. **3**

Example 28

[2020 Independent Ext 1 Trial Q13]

i. Prove the trigonometric identity **2**

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

ii. Hence show that the equation $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$ has roots $\tan\frac{\pi}{9}$, $\tan\frac{4\pi}{9}$ and $\tan\frac{7\pi}{9}$. **3**

iii. Hence show that $\tan^2\frac{\pi}{9} + \tan^2\frac{4\pi}{9} + \tan^2\frac{7\pi}{9} = 33$. **2**

2.3.1 Additional questions

1. **[2020 Hornsby Girls HS Ext 1 Trial Q14]**

i. Prove the trigonometric identity: **3**

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

ii. Using the identity above, show that the roots of the cubic equation **2**

$$x^3 - 3x^2 - 3x + 1 = 0$$

are $\tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$ and -1.

iii. Hence, show that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ **2**

2. (Sadler & Ward, 2019, Ex 3B Q11) - modified

i. It is given that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots

$$x = \pm\tan\frac{\pi}{5} \text{ and } x = \pm\tan\frac{2\pi}{5}$$

ii. Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ and $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$.

3. (Fitzpatrick, 1991, Ex 36(c))

i. Given $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, find an expression for $\cos 4\theta$ in terms of $\cos\theta$ only.

ii. Use the result to show that the equation $8x^4 - 8x^2 + 1 = 0$ has roots

$$x = \cos\frac{\pi}{8}, \quad \cos\frac{3\pi}{8}, \quad \cos\frac{5\pi}{8}, \quad \cos\frac{7\pi}{8}$$

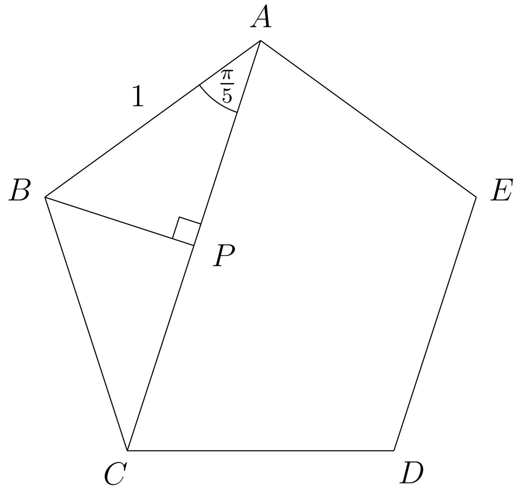
iii. Hence show that

$$\cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0$$

and

$$\cos^2\frac{\pi}{8}\cos^2\frac{3\pi}{8} = \frac{1}{8}$$

4. **[2007 4U HSC Q5]** In the diagram, $ABCDE$ is a regular pentagon with sides of length 1. The perpendicular to AC through B meets AC at P.



i. Let $u = \cos\dfrac{\pi}{5}$. **2**

Use the cosine rule in $\triangle ACD$ to show that $8u^3 - 8u^2 + 1 = 0$.

ii. One root of $8x^3 - 8x^2 + 1 = 0$ is $\dfrac{1}{2}$. **2**

Find the other roots of $8x^3 - 8x^2 + 1 = 0$, and hence find the exact value of $\cos\dfrac{\pi}{5}$.

Section 3

Cosine double angle transformation

Learning Goal(s)

Knowledge
Apply the cosine double angle formulae to obtain expressions for $\sin^2 nx$ and $\cos^2 nx$

Skills
Trigonometric manipulations

Understanding
Where these transformations arise from

By the end of this section am I able to:

26.8 Prove and use the identities $\sin^2 nx = \frac{1}{2} - \frac{1}{2}\cos 2nx$ and $\cos^2 nx = \frac{1}{2} + \frac{1}{2}\cos 2nx$ to solve problems

26.9 Solve problems involving $\int \sin^2 nx\, dx$ and $\int \cos^2 nx\, dx$

Laws/Results

From Section 1.1 on page 5:

$$\cos 2A = 2\cos^2 A - 1 \quad \text{(Variant 2)}$$

$$= 1 - 2\sin^2 A \quad \text{(Variant 3)}$$

Rearrange Variant 2:

$$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$$

Rearrange Variant 3:

$$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$$

Example 29

[1996 3U HSC Q3] (2 marks) Show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$$

Example 30

[2010 Ext 1 HSC Q3] (2 marks) The derivative of a function $f(x)$ is given by

$$f'(x) = \sin^2 x$$

Find $f(x)$, given $f(0) = 2$.

Example 31

[2012 Ext 1 HSC Q7] (1 mark) Which expression is equal to

$$\int \sin^2 3x \, dx$$

(A) $\dfrac{1}{2}\left(x - \dfrac{1}{3}\sin 3x\right) + C$

(B) $\dfrac{1}{2}\left(x + \dfrac{1}{3}\sin 3x\right) + C$

(C) $\dfrac{1}{2}\left(x - \dfrac{1}{6}\sin 6x\right) + C$

(D) $\dfrac{1}{2}\left(x + \dfrac{1}{6}\sin 6x\right) + C$

Example 32

[1996 3U HSC Q5]

i. Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$. **2**

ii. Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$. **2**

iii. Find the area enclosed between the curves $y = \sin 2x$ and $y = 2\sin^2 x$ for $0 \leq x \leq \frac{\pi}{4}$. **2**

Further exercises

Ex 12C (Pender et al., 2019b)

- Q3-12

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

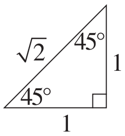
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

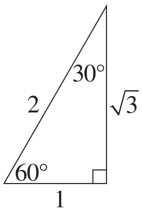
$A = \frac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

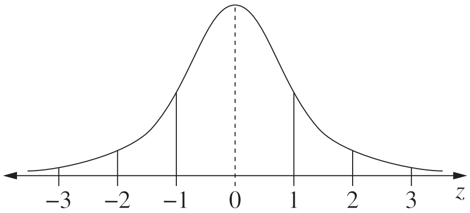
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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